

Itô-Wiener chaos and the Hodge decomposition on an abstract Wiener space

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Abstract: Using the structure of the Boson-Fermion Fock space and an argument taken from [2], we give a new proof of the triviality of the L^2 cohomology groups on an abstract Wiener space, alternative to that given by Shigekawa [9]. We apply some representation theory of the symmetric group to characterise the spaces of exact and co-exact forms in their Boson-Fermion Fock space representation.

Keywords: Wiener chaos, Hodge decomposition, L^2 cohomology, abstract Wiener space, Boson-Fermion Fock space, representation theory, symmetric group.

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1 Introduction

With a Weitzenböck formula for the Hodge Laplacian with positive curvature, I. Shigekawa [9] proved the vanishing of the L^2 de Rham cohomology on an abstract Wiener space. This was a first step in developing a Hodge theory on infinite dimensional manifolds, a goal first set by L. Gross [4] in his pioneering work on infinite dimensional potential theory. Shigekawa's definitive treatment of the linear case provides guidance for the study of nonlinear cases. For example, Fang and Franchi [3] used the Itô map to transfer Shigekawa's results from the Wiener space to the path spaces over a compact Lie group with a bi-invariant metric. However, the problem of developing a Hodge theory in more general infinite-dimensional manifolds remains open.

We present here a different interpretation of Shigekawa's results based on the well-known isometric isomorphism between the L^2 Gaussian space and the symmetric Fock space. This isomorphism is known as the Itô-Wiener chaos expansion [6] on the classical Wiener space, which expresses

any square integrable functional as a sum of multiple stochastic integrals. We extend this isomorphism to skew-symmetric L^2 vector fields, which in turn allows us to transform the study of the Hodge theory on an abstract Wiener space (E, H, μ) to the study of vector subspaces of $H^{\otimes n}$ of the form $H^{\odot k} \otimes H^{\wedge(n-k)}$ for varying integers k . Such Boson-Fermion Fock spaces have been studied by A. Arai [1], among others. Here $H^{\otimes n}$ denotes the completed n -th tensor powers of the Hilbert space H , with $H^{\odot n}$ and $H^{\wedge n}$ its subspaces of symmetric and skew-symmetric tensors, respectively, both completed using the Hilbert space cross norm inherited from $H^{\otimes n}$.

In this context, the de Rham complex of exterior differential forms on our abstract Wiener space can be seen to restrict to two long exact sequences of vector spaces of the form $H^{\odot k} \otimes H^{\wedge(n-k)}$. The exactness of such sequences has been studied in [2] in a purely algebraic setting. The result of [2] involves an identity analogous to Shigekawa's Weitzenböck formula, and implies the Hodge decomposition and the triviality of the de Rham cohomology groups on the abstract Wiener space. Using the representation theory of symmetric groups, we prove further a direct-sum decomposition of the tensor product $H^{\odot k} \otimes H^{\wedge(n-k)}$ considered as a subspace of $H^{\otimes n}$, which shows geometrically how the exact sequences split. Such a decomposition of supersymmetric Fock spaces is of independent interest; algebraically it goes back to W. Hamernik [5] for the finite dimensional case.

The organisation of this article is as follows. After a quick review of the basic notation and Shigakawa's results in Section 2, we explain the Boson-Fermion Fock-space interpretation of these results in Section 3. In Section 4, we present a representation-theoretic proof of the direct-sum decomposition of the tensor product $H^{\odot k} \otimes H^{\wedge(n-k)}$.

2 Notation and Shigekawa's Results

On an abstract Wiener space (E, H, μ) , the natural notion of differentiability is H -differentiability, and correspondingly we consider H -differential-forms, i.e., sections of dual bundle of exterior powers of H . Since we are primarily interested in the L^2 theory, we concentrate on L^2 differential q -forms, denoted $L^2\Gamma(H^{\wedge q})^*$.

The exterior derivative d_q and its adjoint d_q^* are defined, as in Shigekawa [9], by

$$d_q = (q+1)A_{q+1}D, \quad \text{and} \quad d_q^* = D^*,$$

respectively, with D denoting the H -derivative, and D^* its adjoint; all the operators here are closed and densely defined. Note that the standard skew-

symmetrisation operator A_q on q -tensors, given by

$$A_q(h_1 \otimes \cdots \otimes h_q) = \frac{1}{q!} \sum_{\rho \in \mathfrak{S}_q} \text{sgn}(\rho) (h_{\rho(1)} \otimes \cdots \otimes h_{\rho(q)}), \quad h_1, \dots, h_q \in H,$$

is extended to give the alternating map (denoted by the same symbol) on linear functionals defined on tensor products, so given any $\phi \in L(H^{\otimes q}; \mathbb{R})$,

$$A_q \phi(h_1, \dots, h_q) = \frac{1}{q!} \sum_{\rho \in \mathfrak{S}_q} \text{sgn}(\rho) \phi(h_{\rho(1)}, \dots, h_{\rho(q)}).$$

The summation here is over all $q!$ elements of the symmetric group \mathfrak{S}_q , which consists of all permutations of $\{1, \dots, q\}$.

Shigekawa [9] derived the following Weitzenböck identity:

$$\Delta_q = L + q\text{Id}, \quad (1)$$

where $\Delta_q = d_q^* d_q + d_{q-1} d_{q-1}^*$ is the Hodge-Kodaira Laplacian on q -forms, and $L = D^* D$ is the Ornstein-Uhlenbeck operator. Denote by \mathfrak{h}_q the set of all the harmonic forms of degree q , i.e., $\phi \in L^2 \Gamma(\wedge^q H)^*$ is in \mathfrak{h}_q if $\phi \in \text{Dom}(\Delta_q)$ and $\Delta_q \phi = 0$.

Theorem 2.1 (Shigekawa [9]). $L^2 \Gamma(\wedge^q H)^* = \text{Image}(d_{q-1}) \oplus \text{Image}(d_q^*) \oplus \mathfrak{h}_q$, where

1. $\text{Image}(d_{q-1}) = \text{Ker}(d_q)$;
2. $\text{Image}(d_q^*) = \text{Ker}(d_{q-1}^*)$; and
3. $\mathfrak{h}_q = \{0\}$ for $q \geq 1$, and $\mathfrak{h}_0 = \{\text{constant functions}\}$.

Equivalently, the following sequences are exact:

$$0 \rightarrow \mathbb{R} \xrightarrow{\iota} L^2 \Gamma \mathbb{R} \xrightarrow{d_0} L^2 \Gamma H^* \xrightarrow{d_1} \cdots \xrightarrow{d_{q-1}} L^2 \Gamma(H^{\wedge q})^* \xrightarrow{d_q} \cdots$$

and

$$0 \leftarrow \mathbb{R} \xleftarrow{\iota^*} L^2 \Gamma \mathbb{R} \xleftarrow{d_0^*} L^2 \Gamma H^* \xleftarrow{d_1^*} \cdots \xleftarrow{d_{q-1}^*} L^2 \Gamma(H^{\wedge q})^* \xleftarrow{d_q^*} \cdots$$

where the maps $\iota : \mathbb{R} \rightarrow L^2 \Gamma \mathbb{R}$ and $\iota^* : L^2 \Gamma \mathbb{R} \rightarrow \mathbb{R}$ are defined by

$$\iota(c)(x) = c, \quad c \in \mathbb{R}, x \in E,$$

and

$$\iota^* f = \mathbb{E} f, \quad f \in L^2(E; \mathbb{R}).$$

respectively.

3 A Fock-Space Interpretation

We can work with skew-symmetric vector-fields instead of differential forms, which is justified by the Riesz correspondence between the Hilbert space H and its dual H^* , and similarly between the completed tensor powers $H^{\otimes n}$ and its dual $(H^{\otimes n})^*$. The consequent correspondence between L^2 H -forms and skew-symmetric L^2 H -vector fields

$$L^2\Gamma(H^{\wedge q})^* \cong L^2\Gamma H^{\wedge q} \cong L^2(E, \mu; \mathbb{R}) \otimes H^{\wedge q}$$

allows us to define, corresponding to the exterior derivative d_q on q -forms, an operator $d_q^\sharp : \text{Dom}(d_q^\sharp) \subset L^2\Gamma H^{\wedge q} \rightarrow L^2\Gamma H^{\wedge(q+1)}$ on skew-symmetric q -vector fields by

$$d_q^\sharp u_\sigma = (d_q u_\sigma^\sharp)^\sharp, \quad u \in L^2\Gamma H^{\wedge q}, \sigma \in E,$$

where $u^\sharp \in L^2\Gamma(\wedge^q H)^*$ is given by $u_\sigma^\sharp(h) = \langle u_\sigma, h \rangle_{H^{\wedge q}}$, and $u \in \text{Dom}(d_q^\sharp)$ iff $u^\sharp \in \text{Dom}(d_q)$. Similarly for d_q^* , we define an operator $d_q^{*\sharp}$ by

$$d_q^{*\sharp} u_\sigma = (d_q^* u_\sigma^\sharp)^\sharp, \quad u \in L^2\Gamma H^{\wedge(q+1)}, \sigma \in E,$$

where $u \in \text{Dom}(d_q^{*\sharp})$ iff $u^\sharp \in \text{Dom}(d_q^*)$. We note that $d_q^{*\sharp} = d_q^{\sharp*}$, and that all the operators here are closed and densely defined.

We follow Shigekawa [9] to define the exterior product \wedge by

$$h_1 \wedge \cdots \wedge h_q = q! A_q(h_1 \otimes \cdots \otimes h_q), \quad h_1, \dots, h_q \in H,$$

Similarly the symmetric product \odot is defined by

$$h_1 \odot \cdots \odot h_k = k! S_k(h_1 \otimes \cdots \otimes h_k), \quad h_1, \dots, h_k \in H,$$

with the symmetrisation operator S_k on k -tensors given by

$$S_k(h_1 \otimes \cdots \otimes h_k) = \frac{1}{k!} \sum_{\rho \in \mathfrak{S}_k} (h_{\rho(1)} \otimes \cdots \otimes h_{\rho(k)}), \quad h_i \in H.$$

As in Shigekawa [9], we use the following inner product for $H^{\wedge q}$ (instead of the usual inner product induced from $H^{\otimes q}$)

$$\langle h_1 \wedge \cdots \wedge h_q, g_1 \wedge \cdots \wedge g_q \rangle = \det(\langle h_i, g_j \rangle), \quad h_1, \dots, h_q, g_1, \dots, g_q \in H,$$

and similarly for $H^{\odot k}$,

$$\langle h_1 \odot \cdots \odot h_k, g_1 \odot \cdots \odot g_k \rangle = \sum_{\rho \in \mathfrak{S}_k} \prod_{i=1}^k \langle h_i, g_{\rho(i)} \rangle.$$

The well-known isometry between the symmetric (or Boson) Fock space over H , $\mathbf{F}_s(H) = \bigoplus_{k=0}^{\infty} H^{\odot k}$, and the L^2 Gaussian space, $L^2(E, \gamma; \mathbb{R})$,

$$\Psi : \mathbf{F}_s(H) \cong L^2(E, \mu; \mathbb{R}),$$

is defined by (see, e.g., [8])

$$\Psi(\exp \odot h) = \exp(I(h) - \frac{1}{2}\|h\|^2), \quad h \in H, \quad (2)$$

where

$$\exp \odot h = \sum_{k=0}^{\infty} \frac{1}{k!} h^{\odot k} \in \bigoplus_{k=0}^{\infty} H^{\odot k}.$$

Recall that these exponential vectors $\{\exp \odot h\}_{h \in H}$ form a total subset of $\mathbf{F}_s(H)$. The isometry Ψ can be extended to skew-symmetric L^2 H -vector fields to obtain isomorphisms

$$\Psi_q : \mathbf{F}_s(H) \otimes H^{\wedge q} \cong L^2(E, \mu; \mathbb{R}) \otimes H^{\wedge q},$$

if we set, for each $q \in \mathbb{N}$,

$$\Psi_q = \Psi \otimes \text{Id}_{H^{\wedge q}}. \quad (3)$$

To relate to the Hodge theory on our abstract Wiener space, we first observe that, if the following diagram commutes

$$\begin{array}{ccccc} \bigoplus_{k=0}^{\infty} H^{\odot k} \otimes H^{\wedge q} & \xrightarrow{\Psi_q} & L^2(E, \mu; \mathbb{R}) \otimes H^{\wedge q} & \xrightarrow{\cong} & L^2\Gamma(H^{\wedge q})^* \\ \check{d}_q \downarrow & & d_q^\# \downarrow & & d_q \downarrow \\ \bigoplus_{k=1}^{\infty} H^{\odot k-1} \otimes H^{\wedge(q+1)} & \xrightarrow{\Psi_{q+1}} & L^2(E, \mu; \mathbb{R}) \otimes H^{\wedge(q+1)} & \xrightarrow{\cong} & L^2\Gamma(H^{\wedge(q+1)})^*, \end{array} \quad (4)$$

it reduces the study of the exterior derivative d_q and its adjoint d_q^* on the abstract Wiener space (E, H, μ) , which are on the right side of the diagram, to that of their counterparts on the extended Fock spaces, on the left side. We start by defining the operator \check{d}_q on $\mathbf{F}_s(H) \otimes H^{\wedge q}$, and its adjoint operator \check{d}_q^* on $\mathbf{F}_s(H) \otimes H^{\wedge(q+1)}$, via their restrictions on each k -th chaos.

Definition 3.1. Given $k, q \in \mathbb{N}$, and $h_1, \dots, h_k, x_1, \dots, x_q, x_{q+1} \in H$, we set $\check{d}_q(x_1 \wedge \dots \wedge x_q) = 0$,

$$\begin{aligned} & \check{d}_q(h_1 \odot \dots \odot h_k \otimes x_1 \wedge \dots \wedge x_q) \\ &= \sum_{j=1}^k h_1 \odot \dots \odot \hat{h}_j \odot \dots \odot h_k \otimes h_j \wedge x_1 \wedge \dots \wedge x_q, \end{aligned}$$

where $\hat{}$ denotes omission, and

$$\begin{aligned} & \check{d}_q^*(h_1 \odot \cdots \odot h_k \otimes x_1 \wedge \cdots \wedge x_{q+1}) \\ &= \sum_{i=1}^{q+1} (-1)^{i-1} h_1 \odot \cdots \odot h_k \odot x_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_{q+1}. \end{aligned}$$

For brevity, we use the shorthand notation $H_{k,q} = H^{\odot k} \otimes H^{\wedge q}$ for this subspace of $H^{\otimes(k+q)}$. It is clear that, up to constant multiples, the operator \check{d}_q is the composition of the inclusion of $H_{k,q}$ into $H^{\otimes(k+q)}$ followed by the projection onto $H_{k-1,q+1}$. It is not difficult to verify the following properties:

- (a) $\check{d}_{q+1} \circ \check{d}_q = 0$, $\check{d}_q^* \circ \check{d}_{q+1}^* = 0$;
- (b) $\check{d}_q|_{H_{k,q}} = (q+1)\text{Id}_{H^{\odot(k-1)}} \otimes A_{q+1}$, $\check{d}_q(H_{k,q}) \subset H_{k-1,q+1}$; and
- (c) $\check{d}_q^*|_{H_{k-1,q+1}} = kS_k \otimes \text{Id}_{H^{\wedge q}}$, $\check{d}_q^*(H_{k-1,q+1}) \subset H_{k,q}$.

Using (b) and (c), we can check that \check{d}_q and \check{d}_q^* are adjoint to each other.

Lemma 3.2. *The diagram (4) commutes, and*

$$\check{d}_q[(\exp \odot h) \otimes x] = (\exp \odot h) \otimes (h \wedge x), \quad \forall h \in H, x \in H^{\wedge q}. \quad (5)$$

Proof. Given any $h \in H$ and $x \in H^{\wedge q}$, we verify that

$$\begin{aligned} \check{d}_q[(\exp \odot h) \otimes x] &= \check{d}_q\left[\sum_{k=0}^{\infty} \frac{1}{k!} h^{\odot k} \otimes x\right] \\ &= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} h^{\odot(k-1)} \otimes h \wedge x \\ &= (\exp \odot h) \otimes (h \wedge x). \end{aligned}$$

The definitions (2) and (3) give

$$\Psi_q[(\exp \odot h) \otimes x] = \exp(I(h) - \frac{1}{2}\|h\|^2) \otimes x, \quad \forall h \in H, x \in H^{\wedge q},$$

so we have

$$\begin{aligned} d_q^\# \Psi_q[(\exp \odot h) \otimes x] &= d_q^\#[\exp(I(h) - \frac{1}{2}\|h\|^2) \otimes x] \\ &= (q+1)A_{q+1}[\exp(I(h) - \frac{1}{2}\|h\|^2) \otimes h \otimes x] \\ &= \exp(I(h) - \frac{1}{2}\|h\|^2) \otimes (h \wedge x) \\ &= \Psi_{q+1}[(\exp \odot h) \otimes (h \wedge x)] \\ &= \Psi_{q+1} \check{d}_q[(\exp \odot h) \otimes x]. \end{aligned}$$

Therefore, the diagram indeed commutes. \square

To understand how the operators \check{d}_q and \check{d}_q^* interact when they map into different components of $H_{k,q}$ for varying integers k and q , we quote the following result of J. Rawnsley, a version of which appeared in [2] for the case of H being a finite-dimensional vector space. The proof given below was shown to us by J. Rawnsley; it is simple and instructive, and does not depend on the dimension of H . Let's fix k and q , and set $n = k + q$.

Proposition 3.3 (Rawnsley). *The following sequences are exact:*

$$0 \rightarrow H^{\odot n} \xrightarrow{\check{d}_0} \cdots \xrightarrow{\check{d}_{q-1}} H_{k,q} \xrightarrow{\check{d}_q} H_{k-1,q+1} \xrightarrow{\check{d}_{q+1}} \cdots \xrightarrow{\check{d}_n} H^{\wedge n} \rightarrow 0$$

and

$$0 \rightarrow H^{\wedge n} \xrightarrow{\check{d}_n^*} \cdots \xrightarrow{\check{d}_{q+1}^*} H_{k-1,q+1} \xrightarrow{\check{d}_q^*} H_{k,q} \xrightarrow{\check{d}_{q-1}^*} \cdots \xrightarrow{\check{d}_0^*} H^{\odot n} \rightarrow 0.$$

Proof. First recall from Property (a) that $\check{d}_{q+1} \circ \check{d}_q = 0$ and $\check{d}_q^* \circ \check{d}_{q+1}^* = 0$. Given any $h_1, \dots, h_k, x_1, \dots, x_q \in H$, we have

$$\begin{aligned} & \check{d}_q^* \circ \check{d}_q(h_1 \odot \cdots \odot h_k \otimes x_1 \wedge \cdots \wedge x_q) \\ &= \check{d}_q^* \left[\sum_{j=1}^k h_1 \odot \cdots \odot \hat{h}_j \odot \cdots \odot h_k \otimes h_j \wedge x_1 \wedge \cdots \wedge x_q \right] \\ &= \sum_{j=1}^k [h_1 \odot \cdots \odot \hat{h}_j \odot \cdots \odot h_k \odot h_j \otimes x_1 \wedge \cdots \wedge x_q \\ & \quad + \sum_{i=1}^q (-1)^i h_1 \odot \cdots \odot \hat{h}_j \odot \cdots \odot h_k \odot x_i \otimes h_j \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_q], \end{aligned}$$

and

$$\begin{aligned} & \check{d}_{q-1} \circ \check{d}_{q-1}^*(h_1 \odot \cdots \odot h_k \otimes x_1 \wedge \cdots \wedge x_q) \\ &= \check{d}_{q-1} \left[\sum_{i=1}^q (-1)^{i-1} h_1 \odot \cdots \odot h_k \odot x_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_q \right] \\ &= \sum_{i=1}^q (-1)^{i-1} [h_1 \odot \cdots \odot h_k \otimes x_i \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_q \\ & \quad + \sum_{j=1}^k h_1 \odot \cdots \odot \hat{h}_j \odot \cdots \odot h_k \odot x_i \otimes h_j \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_q]. \end{aligned}$$

Therefore, we have the following identity on $H_{k,q}$:

$$\check{d}_q^* \circ \check{d}_q + \check{d}_{q-1} \circ \check{d}_{q-1}^* = (k+q)\text{Id}. \quad (6)$$

So \check{d} and \check{d}^* are invertible on the kernel of each other, proving exactness. \square

Remark 3.4. In particular, Proposition 3.3 shows that \check{d} has closed range.

Remark 3.5. Since Ψ_q 's are isomorphisms for all $q \in \mathbb{N}$, Proposition 3.3 shows that the kernel of d and the kernel of d^* are disjoint, which leads to the vanishing result in Theorem 2.1.

Remark 3.6. It is not difficult to notice the similarity between equation (6) and Shigekawa's Weitzenböck identity (1). Recall that $L = D^*D$ corresponds to the number operators on the Fock space and, when acting on elements of $H_{k,q}$, gives the term $k\text{Id}$ on the right-hand side of (6), which explains the difference between (6) and (1).

A bit more can be said of the kernels and images of \check{d} and \check{d}^* in the Fock space setting; note that the images of \check{d} and \check{d}^* correspond to the exact and co-exact forms, respectively, in the abstract Wiener space. For a fixed k , the exact sequences in Proposition 3.3 can be viewed as the restriction of the two sequences in Theorem 2.1. The following lemma gives a direct-sum decomposition of the tensor product $H^{\odot k} \otimes H^{\wedge q}$, considered as a subspace of $H^{\otimes n}$, which shows how the exact sequences split. But we first need to study a more general subspace of $H^{\otimes n}$. Recall that $H_{k,q} = H^{\odot k} \otimes H^{\wedge q}$ is the subspace of elements of $H^{\otimes n}$ symmetric in the *first* k components and skew-symmetric in the last q components. Denote by $H^{\odot[k],\wedge[q]}$ the closed linear span of elements of $H^{\otimes n}$ symmetric in *any* k components and skew-symmetric in the remaining q components; note that we are not fixing the order of the symmetric and skew-symmetric parts with respect to each other inside the tensor product. We use the following shorthand notation from now on: $H_{k,q}^+ = H_{k,q} \cap H^{\odot[k+1],\wedge[q-1]}$, and $H_{k,q}^- = H_{k,q} \cap H^{\odot[k-1],\wedge[q+1]}$.

Lemma 3.7. *Given $k, q \in \mathbb{N}$, let $n = k + q$. We have the decomposition*

$$H_{k,q} = H_{k,q}^+ \oplus H_{k,q}^-, \quad (7)$$

where the equality is understood to take place inside $H^{\otimes n}$.

Lemma 3.7 is a special case of Corollary 4.3 in the next section, for which a representation-theoretic proof is given. The decomposition (7) gives a concrete description of the kernels and images of \check{d} and \check{d}^* , and, together with Proposition 3.3, allows us to state an analogue of Theorem 2.1:

1. $H_{k,q}^+ = \text{Image}(\check{d}_{q-1}|_{H_{k+1,q-1}}) = \text{Ker}(\check{d}_q|_{H_{k,q}})$;
2. $H_{k,q}^- = \text{Image}(\check{d}_q^*|_{H_{k-1,q+1}}) = \text{Ker}(\check{d}_{q-1}^*|_{H_{k,q}})$; and
3. $\text{Ker}(\check{d}_{q-1}^*|_{H_{k,q}}) \cap \text{Ker}(\check{d}_q|_{H_{k,q}}) = \{0\}$.

It is not difficult to verify that $H_{k,q}^+ \subset \text{Ker}(\check{d}_q|_{H_{k,q}})$, since \check{d}_q is, as shown by Property (b), basically a skew-symmetrisation operator and is annihilated by any element of $H_{k,q}$ symmetric in at least two of its last $(q+1)$ -components. Similarly we see $H_{k,q}^- \subset \text{Ker}(\check{d}_{q-1}^*|_{H_{k,q}})$. Proposition 3.3 proves the disjointness of the kernels of $\check{d}_q|_{H_{k,q}}$ and $\check{d}_{q-1}^*|_{H_{k,q}}$, so indeed $H_{k,q}^+ = \text{Ker}(\check{d}_q|_{H_{k,q}})$, $H_{k,q}^- = \text{Ker}(\check{d}_{q-1}^*|_{H_{k,q}})$, and the rest is clear. The above can be more succinctly described by the following diagram of long and short exact sequences:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \uparrow & & \downarrow \\
& & H_{k+1,q-1}^- & \xrightarrow{\check{d}_{q-1}} & H_{k,q}^+ & & H_{k-1,q+1}^- \\
& & \downarrow & \xleftarrow{\check{d}_{q-1}^*} & \uparrow & & \downarrow \\
0 \longrightarrow & H_{n,0} \longrightarrow & \cdots & H_{k+1,q-1}^- & \xrightarrow{\check{d}_{q-1}} & H_{k,q}^+ & \xrightarrow{\check{d}_q} & H_{k-1,q+1}^- \cdots \longrightarrow & H_{0,n} \longrightarrow 0. \\
& & \downarrow & \xleftarrow{\check{d}_{q-1}^*} & \uparrow & \xleftarrow{\check{d}_q^*} & \downarrow & \\
& & H_{k+1,q-1}^+ & & H_{k,q}^- & \xrightarrow{\check{d}_q} & H_{k-1,q+1}^+ & \\
& & \downarrow & & \uparrow & \xleftarrow{\check{d}_q^*} & \downarrow & \\
& & 0 & & 0 & & 0
\end{array}$$

4 A Representation-Theoretic Proof

To prove Lemma 3.7, we first note that the symmetric group of degree n , \mathfrak{S}_n , acts naturally on $H^{\otimes n}$ by permuting the n components. The vector subspace $H_{k,q}$ is not an \mathfrak{S}_n -invariant subspaces of $H^{\otimes n}$, but $H^{\odot[k], \wedge[q]}$ is.

To give a more specific description of $H^{\odot[k], \wedge[q]}$, let $\mathbf{n} = \{1, 2, \dots, n\}$, and define the set of the k -subsets of \mathbf{n}

$$\mathbf{n}^{[k]} = \{\mathbf{a} \subseteq \mathbf{n} \mid |\mathbf{a}| = k\}.$$

For each $\mathbf{a} \in \mathbf{n}^{[k]}$, denote by \mathbf{a}^c its complement in \mathbf{n} , and by $H^{\odot \mathbf{a}, \wedge \mathbf{a}^c}$ the subspace of elements of $H^{\otimes n}$ symmetric in the chosen components, specified

by \mathbf{a} , and skew-symmetric in the remaining components, specified by \mathbf{a}^c . Similarly we have $H^{\odot\mathbf{a},\otimes\mathbf{a}^c}$, the subspace of elements of $H^{\otimes n}$ only restricted to be symmetric in the chosen components specified by \mathbf{a} , and $H^{\wedge\mathbf{a},\otimes\mathbf{a}^c}$, the subspace of elements of $H^{\otimes n}$ skew-symmetric in the chosen components specified by \mathbf{a} . For example, $H_{k,q} = H^{\odot k} \otimes H^{\wedge q}$ correspond to the choice of $\mathbf{a} = \{1, \dots, k\}$, and hence $\mathbf{a}^c = \{k+1, \dots, n\}$, so in this notation it can be written as $H^{\odot\{1,\dots,k\},\wedge\{k+1,\dots,n\}}$.

The subspace $H^{\odot[k],\wedge[q]}$ is the span of all $C(n,k)$ subspaces $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$, corresponding to the $C(n,k)$ possible choices of \mathbf{a} in $\mathbf{n}^{[k]}$, of elements invariant under a permutation of a specific set of k variables, and anti-invariant under a permutation of the rest. For each choice, say, \mathbf{a} and hence \mathbf{a}^c , of the k and q variables, the action of the corresponding $\mathfrak{S}_k \times \mathfrak{S}_q$ stabilises $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$, while the elements of $\mathfrak{S}_n/(\mathfrak{S}_k \times \mathfrak{S}_q)$ permute the spaces $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$ with different choices of \mathbf{a} 's. The structure of $H^{\odot[k],\wedge[q]}$ is similar to that of the representation induced from $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$, but the spaces $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$ with different choices of \mathbf{a} 's in general can intersect non-trivially, and therefore may not form a direct sum.

Recall the symmetrisation and skew-symmetrisation operators S_n and A_n , which project elements of $H^{\otimes n}$ onto the closed subspaces $H^{\odot n}$ and $H^{\wedge n}$, respectively. Corresponding to an element $\mathbf{a} \in \mathbf{n}^{[k]}$, we define the operator $S_{\mathbf{a}} : H^{\otimes n} \rightarrow H^{\odot\mathbf{a},\otimes\mathbf{a}^c}$, which symmetrises any n -tensor in its k components specified by \mathbf{a} , and the operator $A_{\mathbf{a}} : H^{\otimes n} \rightarrow H^{\wedge\mathbf{a},\otimes\mathbf{a}^c}$, which skew-symmetrises any n -tensor in its k components specified by \mathbf{a} . To be more precise, for $h \in H^{\otimes n}$ and $\rho \in \mathfrak{S}_k$, denote by $\rho^{\mathbf{a}}h$ the element of $H^{\otimes n}$ that has its \mathbf{a} components permuted by ρ and the remaining components fixed, so $S_{\mathbf{a}}$ and $A_{\mathbf{a}}$ are defined, respectively, by

$$S_{\mathbf{a}}h = \frac{1}{k!} \sum_{\rho \in \mathfrak{S}_k} \rho^{\mathbf{a}}h, \quad \text{and} \quad A_{\mathbf{a}}h = \frac{1}{k!} \sum_{\rho \in \mathfrak{S}_k} \text{sgn}(\rho) \rho^{\mathbf{a}}h.$$

From our earlier discussion, it is clear that $H^{\odot\mathbf{a},\otimes\mathbf{a}^c}$ is the image of $H^{\otimes n}$ under $S_{\mathbf{a}}$, $H^{\wedge\mathbf{a},\otimes\mathbf{a}^c}$ is the image under $A_{\mathbf{a}}$, and $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$ is the image under $S_{\mathbf{a}}A_{\mathbf{a}^c}$.

If $\{e_i\}_{i \in \mathbb{N}}$ is a complete orthonormal basis of H , we have correspondingly $\{e_{i_1} \otimes \dots \otimes e_{i_n}\}_{i_1, \dots, i_n=1}^{\infty}$ as a complete orthonormal basis of $H^{\otimes n}$. We can choose a basis of $H^{\odot n}$ whose elements are of the form $e_{i_1} \wedge \dots \wedge e_{i_n}$, with i_1, \dots, i_n all integers. Similarly, we also have a basis of $H^{\wedge n}$ with elements of the form $e_{j_1} \wedge \dots \wedge e_{j_n}$, where j_1, \dots, j_n are *distinct* integers. For $H^{\odot k} \otimes H^{\wedge q}$, we can similarly take basis elements of the form

$$e_{i_1} \wedge \dots \wedge e_{i_k} \otimes e_{j_1} \wedge \dots \wedge e_{j_q}, \quad (8)$$

where each of the indices $i_1, \dots, i_k, j_1, \dots$ and j_q run from 1 to ∞ , and the j 's have to be all distinct.

For $H^{\odot[k], \wedge[q]}$, its basis elements can be chosen almost the same way as in (8), but the positions of the components which are symmetric and those which are skew-symmetric depend on one of the $C(n, k)$ choices from $\mathbf{n}^{[k]}$. A given basis element of $H^{\odot k} \otimes H^{\wedge q}$ of the form (8), say, b , corresponds to two specific collections of basis elements of H , counted with multiplicity:

$$E_{b_i} = \{e_{i_1}, \dots, e_{i_k}\}, \quad \text{and } E_{b_j} = \{e_{j_1}, \dots, e_{j_q}\}.$$

The vector b also corresponds to the element $\mathbf{a}_b = \{1, \dots, k\}$ in $\mathbf{n}^{[k]}$. Similarly, a basis element of $H^{\odot[k], \wedge[q]}$ involves firstly the choice of two collections of basis elements of H , counted with multiplicity, where one set (of k elements specified by the i -indices) forms the symmetric part of the basis element, and the other set (of q distinct elements specified by the j -indices) forms the skew-symmetric part; and secondly the choice of an element in $\mathbf{n}^{[k]}$, for the positioning of the k symmetric components. For $b \in H^{\odot k} \otimes H^{\wedge q}$ as in (8), since the action of \mathfrak{S}_n permutes the n components of b , the orbit O_b of b under the action of \mathfrak{S}_n covers all the $C(n, k)$ possibilities of the positioning. We can therefore enumerate all our basis elements of $H^{\odot[k], \wedge[q]}$ by going through the basis elements of $H^{\odot k} \otimes H^{\wedge q}$ of the form (8) (for our purpose, we don't need to worry about the possible repetitions). Denote by V_{O_b} the span of the vectors in O_b , which is a subspace of $H^{\odot[k], \wedge[q]}$. We have thus proved the following

Lemma 4.1. *Given any $k, q \in \mathbb{N}$, we have*

$$H^{\odot[k], \wedge[q]} = \text{Span} \left(\bigcup_b V_{O_b} \right), \quad (9)$$

where the union is taken over all basis elements of a complete orthonormal basis of $H^{\odot k} \otimes H^{\wedge q}$.

In the sequel, we will often study the basis elements of $H^{\odot[k], \wedge[q]}$ through those of $H^{\odot k} \otimes H^{\wedge q}$, which give easier notation for explicit expressions.

For a fixed basis element b of $H^{\odot[k], \wedge[q]}$, there is a subgroup of \mathfrak{S}_n isomorphic to $\mathfrak{S}_k \times \mathfrak{S}_q$ whose representation on the one-dimensional space spanned by b is $[k] \sharp [1^q]$, the outer tensor product of $[k]$ and $[1^q]$, an irreducible representation of $\mathfrak{S}_k \times \mathfrak{S}_q$ (e.g., see Section 2.3 of [7]). The irreducible representation $[k] \sharp [1^q]$ on $\text{Span}(b)$ induces into \mathfrak{S}_n the representation $[k][1^q]$ on V_{O_b} . A simple application of the Littlewood-Richardson rule (Theorem 2.8.13, or

more directly, Corollary 2.8.14, of [7]) yields the following decomposition into irreducible constituents:

$$[k][1^q] = [k+1, 1^{q-1}] \oplus [k, 1^q]. \quad (10)$$

Hence, every subspace V_{O_b} splits into a direct sum of two irreducible components

$$V_{O_b} = V_{O_b}^+ \oplus V_{O_b}^-, \quad (11)$$

where $V_{O_b}^+$ and $V_{O_b}^-$ correspond to $[k+1, 1^{q-1}]$ and $[k, 1^q]$, respectively.

Suppose we have another basis element b' of $H^{\odot[k], \wedge[q]}$, with a corresponding orbit $O_{b'}$ and an associated subspace $V_{O_{b'}}$. As in the discussion earlier, in terms of the basis elements of H appearing in the expression of b and b' , we have two sets $E_{b_i} = \{e_{i_1}, \dots, e_{i_k}\}$ and $E_{b_j} = \{e_{j_1}, \dots, e_{j_q}\}$, where the i 's and j 's are integers and the j 's are all distinct; and similarly $E_{b'_i} = \{e_{i'_1}, \dots, e_{i'_k}\}$ and $E_{b'_j} = \{e_{j'_1}, \dots, e_{j'_q}\}$, where the i' 's and j' 's are integers and the j' 's are all distinct.

Observe that the orbits O_b and $O_{b'}$ are disjoint and the spaces V_{O_b} and $V_{O_{b'}}$ have a trivial intersection, as long as the sequences (E_{b_i}, E_{b_j}) and $(E_{b'_i}, E_{b'_j})$ differ. Therefore, each V_{O_b} intersects at most finitely many other subspaces $V_{O_{b'}}$. If we have a non-trivial element $v \in V_{O_b} \cap V_{O_{b'}}$, the orbit of v under the action of \mathfrak{S}_n spans an invariant subspace of both V_{O_b} and $V_{O_{b'}}$. Our earlier discussion shows that, either these two spaces coincide, i.e.,

$$V_{O_b} = V_{O_{b'}} (= V_{O_b} \cap V_{O_{b'}}),$$

or their intersection corresponds to one of the two components in (10), i.e., $[k+1, 1^{q-1}]$ and $[k, 1^q]$, so in terms of (11),

$$V_{O_b} \cap V_{O_{b'}} = V_{O_b}^+ \text{ or } V_{O_b}^-.$$

In summary, the action of \mathfrak{S}_n splits the collection of our basis elements of $H^{\odot[k], \wedge[q]}$ into disjoint subsets, each of which spans a vector subspace of $H^{\odot[k], \wedge[q]}$, which is a copy of the representation $[k][1^q]$ of \mathfrak{S}_n . Any non-trivial intersection of these vector subspaces, when they do not coincide, is limited to be one of the two irreducible components, as shown above. Therefore, the space $H^{\odot[k], \wedge[q]}$ is made of infinitely many finite-dimensional isomorphic representations of \mathfrak{S}_n , each isomorphic to $[k][1^q]$, mostly disjoint from the rest, but possibly intersecting a few along its irreducible components.

This discussion enables us to state the following decomposition of $H^{\odot[k], \wedge[q]}$ inside $H^{\otimes n}$.

Lemma 4.2. *Given any $k, q \in \mathbb{N}$, let $n = k + q$. Then we have*

$$H^{\odot[k], \wedge[q]} = (H^{\odot[k], \wedge[q]} \cap H^{\odot[k+1], \wedge[q-1]}) \oplus (H^{\odot[k], \wedge[q]} \cap H^{\odot[k-1], \wedge[q+1]}), \quad (12)$$

where the equality is understood to take place inside $H^{\otimes n}$.

Proof. As mentioned above, \mathfrak{S}_n acts naturally on $H^{\otimes n}$ by permuting the n components. The vector subspaces in question, i.e., $H^{\odot[k], \wedge[q]}$, $H^{\odot[k+1], \wedge[q-1]}$, and $H^{\odot[k-1], \wedge[q+1]}$, as well as their intersections appearing in (12), are \mathfrak{S}_n -invariant subspaces of $H^{\otimes n}$.

Lemma 4.1 and the discussion afterwards show that $H^{\odot[k], \wedge[q]}$ consists of subspaces isomorphic to the representation $[k][1^q]$ of \mathfrak{S}_n , each intersecting finitely many others, with the non-trivial intersection being one of the two irreducible components of $[k][1^q]$. The same statements can be made for $H^{\odot[k+1], \wedge[q-1]}$ and $H^{\odot[k-1], \wedge[q+1]}$, but replacing $[k][1^q]$ with $[k+1][1^{q-1}]$ and $[k-1][1^{q+1}]$, respectively.

Similar to (10), we also have the Littlewood-Richardson decompositions for $[k+1][1^{q-1}]$ and $[k-1][1^{q+1}]$, i.e.,

$$[k+1][1^{q-1}] = [k+2, 1^{q-2}] \oplus [k+1, 1^{q-1}]$$

and

$$[k-1][1^{q+1}] = [k, 1^q] \oplus [k-1, 1^{q+1}],$$

respectively. Observe that $[k][1^q]$ has exactly one irreducible component in common with $[k+1][1^{q-1}]$, which is $[k+1, 1^{q-1}]$, and exactly one with $[k-1][1^{q+1}]$, which is $[k-1, 1^{q+1}]$, and no other ones.

Now Lemma 4.1 implies that $H^{\odot[k+1], \wedge[q-1]}$ and $H^{\odot[k-1], \wedge[q+1]}$ have only a trivial intersection; indeed, the intersection would have to be \mathfrak{S}_n -invariant, but (9) and (10) show that it has to be trivial. Therefore, we only need to show that $H^{\odot[k], \wedge[q]}$ does intersect $H^{\odot[k+1], \wedge[q-1]}$ and $H^{\odot[k-1], \wedge[q+1]}$ separately, in a manner corresponding to the way $[k][1^q]$ intersects with $[k+1][1^{q-1}]$ and $[k-1][1^{q+1}]$, which then gives us the direct sum as in (12).

Again we can look at an arbitrary basis element b of the form (8), and its associated vector subspace $V_{O_b} \subset H^{\odot[k], \wedge[q]}$. All we need is to find two vectors,

$$v^+ \in V_{O_b} \cap H^{\odot[k+1], \wedge[q-1]}, \quad \text{and } v^- \in V_{O_b} \cap H^{\odot[k-1], \wedge[q+1]},$$

since the two disjoint invariant subspaces of V_{O_b} , $V_{O_b} \cap H^{\odot[k+1], \wedge[q-1]}$ and $V_{O_b} \cap H^{\odot[k-1], \wedge[q+1]}$, have to correspond to the $[k+1, 1^{q-1}]$ and $[k, 1^q]$ components, respectively.

Denote by $\tau_{i,j}$ the transposition operator on $H^{\otimes n}$, which acts by exchanging the i -th and j -th components of a tensor product: i.e., given any $h_1, \dots, h_n \in H$,

$$\tau_{i,j}(h_1 \otimes \dots \otimes h_i \otimes \dots \otimes h_j \otimes \dots \otimes h_n) = h_1 \otimes \dots \otimes h_j \otimes \dots \otimes h_i \otimes \dots \otimes h_n.$$

Now we can express the result of swapping the l -th and $(k+1)$ -th components of b as $\tau_{l,k+1}b$, which is an element of O_b and of $H^{\odot \mathbf{a}(l), \wedge \mathbf{a}(l)^c}$, where we define $\mathbf{a}(l) = \{1, \dots, \hat{l}, \dots, k, k+1\} \in \mathbf{n}^{[k]}$, and l ranges from 1 to $k+1$. Similarly, for each $m = k+1, \dots, n$, we have $\tau_{k,m}b$, an element of O_b and of $H^{\odot \mathbf{a} \langle m \rangle, \wedge \mathbf{a} \langle m \rangle^c}$, with $\mathbf{a} \langle m \rangle = \{1, \dots, k-1, m\} \in \mathbf{n}^{[k]}$. We conclude the proof by setting

$$v^+ = \frac{1}{k+1} \sum_{l=1}^{k+1} \tau_{l,k+1}b$$

and

$$v^- = \frac{1}{q+1} (b - \sum_{m=k+1}^n \tau_{k,m}b). \quad \square$$

Corollary 4.3. *Given any $\mathbf{a} \in \mathbf{n}^{[k]}$, we have*

$$H^{\odot \mathbf{a}, \wedge \mathbf{a}^c} = (H^{\odot \mathbf{a}, \wedge \mathbf{a}^c} \cap H^{\odot [k+1], \wedge [q-1]}) \oplus (H^{\odot \mathbf{a}, \wedge \mathbf{a}^c} \cap H^{\odot [k-1], \wedge [q+1]}). \quad (13)$$

Proof. For any $g \in H^{\odot \mathbf{a}, \wedge \mathbf{a}^c} \subset H^{\odot [k], \wedge [q]}$, we have

$$g = A_{\mathbf{a}^c} S_{\mathbf{a}} g.$$

Lemma 4.2 gives a direct-sum decomposition for g

$$g = \tilde{g} + \hat{g},$$

with $\tilde{g} \in H^{\odot [k], \wedge [q]} \cap H^{\odot [k+1], \wedge [q-1]}$, and $\hat{g} \in H^{\odot [k], \wedge [q]} \cap H^{\odot [k-1], \wedge [q+1]}$. So we have

$$g = A_{\mathbf{a}^c} S_{\mathbf{a}} g = A_{\mathbf{a}^c} S_{\mathbf{a}} \tilde{g} + A_{\mathbf{a}^c} S_{\mathbf{a}} \hat{g},$$

where

$$A_{\mathbf{a}^c} S_{\mathbf{a}} \tilde{g} \in (H^{\odot [k], \wedge [q]} \cap H^{\odot [k+1], \wedge [q-1]}) \cap H^{\odot \mathbf{a}, \wedge \mathbf{a}^c} = H^{\odot \mathbf{a}, \wedge \mathbf{a}^c} \cap H^{\odot [k+1], \wedge [q-1]},$$

and

$$A_{\mathbf{a}^c} S_{\mathbf{a}} \hat{g} \in (H^{\odot [k], \wedge [q]} \cap H^{\odot [k-1], \wedge [q+1]}) \cap H^{\odot \mathbf{a}, \wedge \mathbf{a}^c} = H^{\odot \mathbf{a}, \wedge \mathbf{a}^c} \cap H^{\odot [k-1], \wedge [q+1]}.$$

By the uniqueness of the direct-sum decomposition, we have $\tilde{g} = A_{\mathbf{a}^c} S_{\mathbf{a}} \tilde{g}$ and $\hat{g} = A_{\mathbf{a}^c} S_{\mathbf{a}} \hat{g}$, and the proof is complete. \square

W. Hamernik [5] gave a proof of the decomposition (12) for the case where H is a finite-dimensional vector space.

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